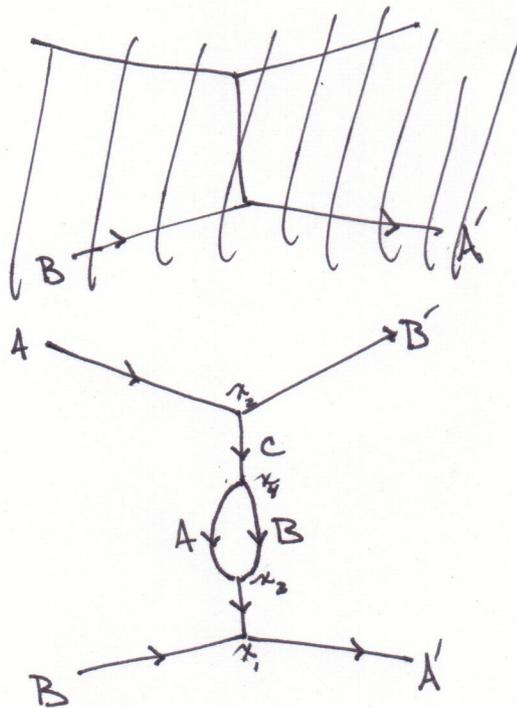


## 10.1 The Propagator Correction in ABC Theory

All processes considered so far have been "tree level." All higher-order processes beyond the tree approximation involve loops. This chapter will focus on the following particular loop:



As there are 4 vertices in this diagram, we are considering a 4<sup>th</sup> order process. The following will be analogous to the "A+B → A+B" scattering calculation for  $S_{fi}^{(2)}$ , the 2<sup>nd</sup> order transition amplitude.

$$S_{fi}^{(4)} = \frac{(-ig)^4}{4!} \iiint d^4x_1 d^4x_2 d^4x_3 d^4x_4 \langle P_{A'}, P_{B'} | T [\mathcal{H}(x_1) \mathcal{H}(x_2) \mathcal{H}(x_3) \mathcal{H}(x_4)] | P_A, P_B \rangle$$

$$S_{fi}^{(4)} = \frac{(-ig)^4}{4!} \iiint d^4x_1 d^4x_2 d^4x_3 d^4x_4 \langle 0 | a_A(p'_A) a_B(p'_B)$$

$$\cdot T [\phi_A(x_1) \phi_B(x_2) \phi_C(x_3) \dots \phi_A(x_4) \phi_B(x_4) \phi(x_4)]$$

$$\cdot a_A^\dagger(p_A) a_B^\dagger(p_B) |0\rangle \sqrt{16 E_A E_B E_{A'} E_{B'}}$$

As this is a  $u$ -channel process, the analogous tree level v.e.v. is given by (6.89). Remember that operators for different particles commute while operators for the same type of particle do not commute, in general. As such  $\phi_A(x_1)$  can move next to  $a_A(p'_A)$ .  $\phi_B(x_2)$  moves next to  $a_B(p'_B)$ .  $\phi_A(x_3)$  moves next to  $a_A^\dagger(p_A)$ .  $\phi_B(x_4)$  moves next to  $a_B^\dagger(p_B)$ .

$$S_{fi}^{(4)} = \frac{(-ig)^4}{4!} \iiint d^4x_1 d^4x_2 d^4x_3 d^4x_4 \langle 0 | a_A(p'_A) \phi_A(x_1) a_B(p'_B) \phi_B(x_2)$$

$$\cdot T [\phi_C(x_3) \phi_C(x_4) \phi_A(x_3) \phi_B(x_3) \phi_C(x_3) \phi_A(x_4) \phi_B(x_4) \phi_C(x_4)]$$

$$\cdot \phi_A(x_2) a_A^\dagger(p_A) \phi_B(x_1) a_B^\dagger(p_B) |0\rangle \sqrt{16 E_A E_B E_{A'} E_{B'}}$$

$$\begin{aligned}
 S_{fi}^{(4)} &= \frac{(-ig)^4}{4!} \iiint d^4x_1 d^4x_2 d^4x_3 d^4x_4 \langle 0 | a_A(p'_A) \phi_A(x_1) | 0 \rangle \langle 0 | a_B(p'_B) \phi_B(x_2) | 0 \rangle \\
 &\cdot \langle 0 | T [\phi_C(x_1) \phi_C(x_2) \phi_A(x_3) \phi_B(x_3) \phi_C(x_3) \phi_A(x_4) \phi_B(x_4) \phi_C(x_4)] | 0 \rangle \\
 &\cdot \langle 0 | \phi_A(x_2) a_A^\dagger(p_A) | 0 \rangle \langle 0 | \phi_B(x_1) a_B^\dagger(p_B) | 0 \rangle \sqrt{16 E_A E_B E'_A E'_B}
 \end{aligned}$$

$$\begin{aligned}
 S_{fi}^{(4)} &= (-ig)^4 \iiint d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i(p'_A - p_B) \cdot x_1} e^{i(p'_B - p_A) \cdot x_2} \\
 &\cdot \langle 0 | T [\phi_C(x_1) \phi_C(x_2) \phi_A(x_3) \phi_B(x_3) \phi_C(x_3) \phi_A(x_4) \phi_B(x_4) \phi_C(x_4)] | 0 \rangle
 \end{aligned}$$

We still have a v.e.v. of 8 operators. As there are many ways to permute 8 terms, we pick out the one appropriate for the u-channel process under consideration.

$$\begin{aligned}
 S_{fi}^{(4)} &= (-ig)^4 \iiint d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i(p'_A - p_B) \cdot x_1} e^{i(p'_B - p_A) \cdot x_2} \\
 &\cdot \langle 0 | T [\phi_C(x_1) \phi_C(x_3)] | 0 \rangle \langle 0 | T [\phi_C(x_2) \phi_C(x_4)] | 0 \rangle \\
 &\cdot \langle 0 | T [\phi_A(x_3) \phi_A(x_4)] | 0 \rangle \langle 0 | T [\phi_B(x_3) \phi_B(x_4)] | 0 \rangle
 \end{aligned}$$

## Problem 10.1

To progress further, we introduce a coordinate transformation to ease calculations. The new coordinates will be defined by the spacetime intervals over which internal particles traverse, and one of the coordinates will be the center of momentum coordinate. Later on, this will result in an integral that will become a  $\delta$ -function that enforces conservation of 4-momentum.

$$x = x_1 - x_3$$

$$y = x_2 - x_4$$

$$z = x_3 - x_4$$

$$X = \frac{1}{4}(x_1 + x_2 + x_3 + x_4)$$

For this transformation to be useful, the coordinates must be inverted to make the substitution.

$$\begin{bmatrix} x \\ y \\ z \\ X \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

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$$\left[ \begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 1 \end{array} \right]$$

Multiply row 4 by 4

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 4 \end{array} \right]$$

Add (-1) times row 1 to row 4.

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 & 0 & 4 \end{array} \right]$$

Add (-1) times row 2 to row 4.

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & -1 & -1 & 0 & 4 \end{array} \right]$$

Add row 3 to row 1.

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & -1 & -1 & 0 & 4 \end{array} \right]$$

Add  $(-2)$  times row 3 to row 4.

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & -1 & -1 & -2 & 4 \end{array} \right]$$

~~And~~ Multiply row 4 by  $\frac{1}{4}$ .

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & 1 \end{array} \right]$$

Add row 4 to row 1.

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & 1 \end{array} \right]$$

Add row 4 to row 2.

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & 1 \end{array} \right]$$

Add row 4 to row 3.

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & 1 \end{array} \right]$$

Finally the inverse may be written as

$$\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & \frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ X \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

I'm not sure how the differentials go from  $dx_1 dx_2 dx_3 dx_4 \rightarrow dx dy dz dX$ , but the rest is now straightforward.

$$\begin{aligned} S_{fi}^{(4)} &= (-ig)^4 \iiint dX dx dy dz e^{i(p'_A + p'_B - p_A - p_B) \cdot X} e^{i(p'_A - p_B) \cdot (3x - y + 2z) \frac{1}{4}} \\ &\quad e^{i(p'_B - p_A) \cdot (-x + 3y - 2z) \frac{1}{4}} \langle 0 | T [\phi_c(x_1) \phi_c(x_3)] | 0 \rangle \\ &\quad \langle 0 | T [\phi_c(x_2) \phi_c(x_4)] | 0 \rangle \langle 0 | T [\phi_A(x_3) \phi_A(x_4)] | 0 \rangle \langle 0 | T [\phi_B(x_3) \phi_B(x_4)] | 0 \rangle \end{aligned}$$

First, note how the integral over the center of momentum coordinate  $X$  enforces conservation of 4-momentum. Second, the ABC Theory propagator has already been calculated so see the appropriate tab in this binder.

$$S_{fi}^{(4)} = (-ig)^4 \iiint d^4x d^4y d^4z (2\pi)^4 \delta^4(\vec{p}'_A + \vec{p}'_B - \vec{p}_A - \vec{p}_B) e^{i(\vec{p}'_A - \vec{p}_B) \cdot (3x - y + 2z) \frac{1}{4}}$$

$$e^{i(\vec{p}'_B - \vec{p}_A) \cdot (-x + 3y - 2z) \frac{1}{4}} \langle 0 | T [\phi_c(x_1) \phi_c(x_2)] | 0 \rangle$$

$$\langle 0 | T [\phi_c(x_2) \phi_c(x_4)] | 0 \rangle \langle 0 | T [\phi_A(x_3) \phi_A(x_4)] | 0 \rangle \langle 0 | T [\phi_B(x_3) \phi_B(x_4)] | 0 \rangle$$

Conservation of 4-momentum ensures we can set  $q \equiv \vec{p}'_A - \vec{p}'_B = \vec{p}'_A - \vec{p}_B$ .

$$S_{fi}^{(4)} = (-ig)^4 (2\pi)^4 \delta^4(\vec{p}'_A + \vec{p}'_B - \vec{p}_A - \vec{p}_B) \iiint d^4x d^4y d^4z e^{ig \cdot \frac{3}{4}x} e^{ig \cdot \frac{1}{4}y} e^{ig \cdot \frac{1}{2}z}$$

$$e^{-ig \cdot \frac{1}{4}x} e^{-ig \cdot \frac{3}{4}y} e^{-ig \cdot \frac{1}{2}z} \langle 0 | T [\phi_c(x_1) \phi_c(x_2)] | 0 \rangle \langle 0 | T [\phi_c(x_2) \phi_c(x_4)] | 0 \rangle$$

$$\langle 0 | T [\phi_A(x_3) \phi_A(x_4)] | 0 \rangle \langle 0 | T [\phi_B(x_3) \phi_B(x_4)] | 0 \rangle$$

$$S_{fi}^{(4)} = (-ig)^4 (2\pi)^4 \delta^4(\vec{p}'_A + \vec{p}'_B - \vec{p}_A - \vec{p}_B) \iiint d^4x d^4y d^4z e^{ig \cdot x} e^{ig \cdot y} e^{ig \cdot z}$$

$$\langle 0 | T [\phi_c(x_1) \phi_c(x_3)] | 0 \rangle \langle 0 | T [\phi_c(x_2) \phi_c(x_4)] | 0 \rangle$$

$$\langle 0 | T [\phi_A(x_3) \phi_A(x_4)] | 0 \rangle \langle 0 | T [\phi_B(x_3) \phi_B(x_4)] | 0 \rangle d^4x d^4y d^4z$$

$$\int_{fi}^{(4)} = (-ig)^4 (2\pi)^4 \delta^4(p'_A + p'_B - p_A - p_B) \int d^4x e^{ig \cdot x} \langle 0 | T [\phi_c(x_1) \phi_c(x_3)] | 0 \rangle$$

$$\cdot \int d^4y e^{ig \cdot y} \langle 0 | T [\phi_c(x_2) \phi_c(x_4)] | 0 \rangle$$

$$\cdot \int d^4z e^{ig \cdot z} \langle 0 | T [\phi_A(x_3) \phi_A(x_4)] | 0 \rangle \langle 0 | T [\phi_B(x_3) \phi_B(x_4)] | 0 \rangle$$

$$\int_{fi}^{(4)} = (-ig)^4 (2\pi)^4 \delta^4(p'_A + p'_B - p_A - p_B) \left[ \frac{i}{q^2 - m_c^2 + i\epsilon} \right]^2$$

$$\cdot \int d^4z e^{ig \cdot z} \langle 0 | T [\phi_A(x_3) \phi_A(x_4)] | 0 \rangle \langle 0 | T [\phi_B(x_3) \phi_B(x_4)] | 0 \rangle$$

Let's look specifically at that last integral:

$$\int d^4z e^{ig \cdot z} \int \frac{d^4k_1}{(2\pi)^4} e^{-ik_1 \cdot z} \frac{i}{k_1^2 - m_A^2 + i\epsilon} \int \frac{d^4k_2}{(2\pi)^4} e^{-ik_2 \cdot z} \frac{i}{k_2^2 - m_B^2 + i\epsilon}$$

$$= \iiint d^4z \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} e^{i(q - k_1 - k_2) \cdot z} \frac{i}{k_1^2 - m_A^2 + i\epsilon} \cdot \frac{i}{k_2^2 - m_B^2 + i\epsilon}$$

$$= (2\pi)^4 \delta^4(k_1 + k_2 - q) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2 - m_A^2 + i\epsilon} \frac{i}{k_2^2 - m_B^2 + i\epsilon} \int d^4z e^{i(k_1 + k_2 - q) \cdot z}$$

$$= (-ig)^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{i}{k_1^2 - m_A^2 + i\epsilon} \cdot \frac{i}{(q - k_1)^2 - m_B^2 + i\epsilon}$$

Drop the subscript on  $k_1$ .

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_A^2 + i\epsilon} \cdot \frac{i}{(q - k)^2 - m_B^2 + i\epsilon}$$

At last, we have the "loop amplitude," and introduce the definition

$$-i \Pi_c^{[2]}(q^2) \equiv (-ig)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m_A^2 + i\epsilon} \cdot \frac{i}{(q-k)^2 - m_B^2 + i\epsilon}$$

And the final amplitude is

$$S_{fi}^{(4)} = (-ig)^2 (2\pi)^4 \delta^4(p'_A + p'_B - p_A - p_B) \frac{i}{q^2 - m_c^2 + i\epsilon} \left( -i \Pi_c^{[2]}(q^2) \right) \frac{i}{q^2 - m_c^2 + i\epsilon}$$

Unfortunately, the integral in  $\Pi$  is divergent. What was to be a small perturbative correction is infinite. This will be dealt with later.